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Stability of some low-order approximations for the Stokes problem

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SUMMARY

Two-level low-order finite element approximations are considered for the inhomogeneous Stokes equations. The elements introduced are attractive because of their simplicity and computational efficiency. In this paper, the stability of a $Q_1(h)$–$Q_1(2h)$ approximation is analysed for general geometries. Using the macroelement technique, we prove the stability condition for both two- and three-dimensional problems. As a result, optimal rates of convergence are found for the velocity and pressure approximations. Numerical results for three test problems are presented. We observe that for the computed examples, the accuracy of the two-level bilinear approximation is compared favourably with some standard finite elements.

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KEY WORDS: Stokes equations; finite elements; stability; error estimates

1. INTRODUCTION

In this paper, we discuss the stability of some low-order finite elements for the Stokes problem. Hence, we will, without loss of generality, consider the Stokes equations with inhomogeneous Dirichlet boundary conditions:

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma \end{aligned} \tag{1}$$

where $\mathbf{u}$, p, ν and $\mathbf{g}$ are, respectively, velocity, pressure, kinematic viscosity (positive and constant), and body force, all of which are assumed to be nondimensionalized.

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We assume that Ω is an open bounded connected subset of R^d, $d=2$ or 3, with smooth boundary Γ. Further, we assume that $\mathbf{f}\in(H^{-1}(\Omega))^d$, $\mathbf{g}\in(H^{1/2}(\Gamma))^d$, and the boundary condition satisfies $\int_\Gamma \mathbf{g}\cdot\mathbf{n}\,\mathrm{d}s=0$. Where, $H^{-1}(\Omega)$ and $H^{1/2}(\Gamma)$ are the usual Sobolev spaces, and $\mathbf{n}$ denotes the outward normal.

A mixed finite element approximation of (1) leads to the discrete problem. Find $\mathbf{u}_h\in\mathbf{V}_h$ and $p_h\in Q_h$ such that

$$\begin{aligned} a(\mathbf{u}_h,\mathbf{v}_h)+b(\mathbf{v}_h,p_h)&=(\mathbf{f},\mathbf{v}_h) \qquad \forall \mathbf{v}_h\in\mathbf{V}_h \\ b(\mathbf{u}_h,q_h)&=0 \qquad \forall q_h\in Q_h \end{aligned} \tag{2}$$

where the bilinear forms $a(\mathbf{u}_h,\mathbf{v}_h)$ and $b(\mathbf{u}_h,q_h)$ are defined by

$$a(\mathbf{u}_h,\mathbf{v}_h)=\int_\Omega \nabla\mathbf{u}_h\cdot\nabla\mathbf{v}_h\,\mathrm{d}\mathbf{x} \quad\text{and}\quad b(\mathbf{u}_h,q_h)=-\int_\Omega q_h\,\mathrm{div}\,\mathbf{u}_h\,\mathrm{d}\mathbf{x} \tag{3}$$

Stable and accurate solution of (2) requires that $\mathbf{V}_h$ and Q_h satisfy the inf–sup condition:

$$\operatorname*{Inf}_{\substack{q_h\in Q_h\\ q_h\neq 0}}\ \operatorname*{Sup}_{\substack{\mathbf{v}_h\in\mathbf{V}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{\int_\Omega q_h\,\mathrm{div}\,\mathbf{v}_h\,\mathrm{d}\mathbf{x}}{\|q_h\|_Q\|\mathbf{v}_h\|_\mathbf{V}}\geqslant\beta \tag{4}$$

where $\beta>0$ is independent of the mesh parameter h; see [1, 2].

To circumvent this stability constraint, stabilized formulations which consist in modifying the weak formulation by including some mesh-dependent terms such as jumps of the pressures across boundary elements were proposed, see [3, 4]. More recently, methods based on pressure stabilization have also been analysed in [5–9].

In this paper, we prove the stability of $Q_1(h)$–$Q_1(2h)$ approximations for general quadrilaterals and hexahedra introduced in [10]. Even though, the stability of the two-level bilinear approximation discussed in this paper can be proved using the Verfurth trick combined with an inf–sup like condition [11, pp. 255–256], the proof given here is more general and a lot simpler. In [12], the tridimensional Taylor–Hood approximation $Q_2(h)$–$Q_1(h)$ has been analysed and it is shown that this approximation is stable. To stabilize nonconstant pressure components, though, the use of quite large macroelements (assuming the existence of neighbouring elements in a required direction) and a suitable high-order quadrature rule were necessary. The analysis given here for the stability of a two-level continuous trilinear approximation, to our knowledge, is new and represents an important step in trying to justify the use of the proposed approximation for solving fluid flow problems. Since both velocity and pressure are bilinear (respectively trilinear), it is shown that macroelement containing only two elements (respectively four elements) of the original mesh ζ_{2h} are needed to stabilize the nonconstant pressure components, and the stability is achieved using composite Simpson's rule. Below, we review briefly the macroelement technique and state the main theorem. In Section 3, we analyse a bilinear approximation for quadrilateral elements and show that it is stable. In Section 4, we extend our analysis to the trilinear approximation and again show that the proposed approximation is stable. Hence, optimal rates of convergence are obtained in the norms of the spaces $[H^1(\Omega)]^d$ and $L^2(\Omega)$. In Section 5, the numerical results for three 2d test problems are presented and the performance of the bilinear approximation is discussed.

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2. STABILITY ANALYSIS

Suppose we have a pair of finite element spaces $\mathbf{V}_h$ and Q_h. We define a macroelement $M \in M_h$ as a union of several neighbouring elements of a finite element partition ζ_h, satisfying the usual regularity assumptions [13]. Further, for a macroelement $M \in M_h$, we define the following local spaces:

$$\begin{aligned} \mathbf{V}_{0,M} &= \mathbf{V}_h \cap (H_0^1(M))^d, \quad Q_M = \{q_{|M} \mid q \in Q_h\} \\ N_M &= \{q \in Q_M \mid (\operatorname{div} \mathbf{v}, q) = 0, \ \forall \mathbf{v} \in \mathbf{V}_{0,M}\} \end{aligned} \tag{5}$$

Denote by Γ_h the set of sides (respectively the set of faces in a three-dimensional space), of the elements of ζ_h interior to Ω. Then, we can prove the following theorem [14].

Theorem 1
Let M_h be a macroelement partition of the elements of ζ_h such that

H1. Each $M \in M_h$ is equivalent to one class ξ_i, $i = \overline{1, r}$, of a fixed set of macroelements.
H2. For each $M \in \xi_i$, $i = \overline{1, r}$, the space N_M is one dimensional consisting of functions that are constant on M.
H3. Each $K \in \zeta_h$ is contained in a finite number of macroelements.
H4. Each $T \in \Gamma_h$ is contained in a finite number of macroelements.

Then, the approximate spaces $\mathbf{V}_h$ and Q_h satisfy stability condition (4).

Remark 2
In general, the main effort to fulfil the hypotheses of the above theorem is to verify H2. In fact, in practice H1, H3, and H4 are met whenever the triangulation ζ_h is regular and the macroelements are reasonable.

We note that nonoverlapping macroelement partitions are not covered by the previous theorem because of hypothesis H4. However, in [15, 16] a similar result was proved.

3. BILINEAR APPROXIMATION

We let ζ_{2h} be a regular partition of Ω into convex quadrilaterals and denote by ζ_h the partition obtained from ζ_{2h} by subdividing each element K into four quadrilaterals. We then define the approximate spaces

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v} \in (H^1(\Omega) \cap C(\overline{\Omega}))^2 | \mathbf{v}_{|K} \in (Q_1(K))^2, \forall K \in \zeta_h\} \\ Q_h &= \{q \in L_0^2(\Omega) \cap C(\overline{\Omega}) | q_{|K} \in Q_1(K), \forall K \in \zeta_{2h}\} \end{aligned} \tag{6}$$

where the corresponding degrees of freedom, given on a reference element, are sketched in Figure 1, and $C(\overline{\Omega})$ denotes the set of functions that are continuous on $\overline{\Omega}$ (closure of Ω). Note that this partition leads to one class of equivalent macroelements, and the hypotheses H1, H3, and H4 hold. Hence, to show that the above approximation is stable, we need to check H2.

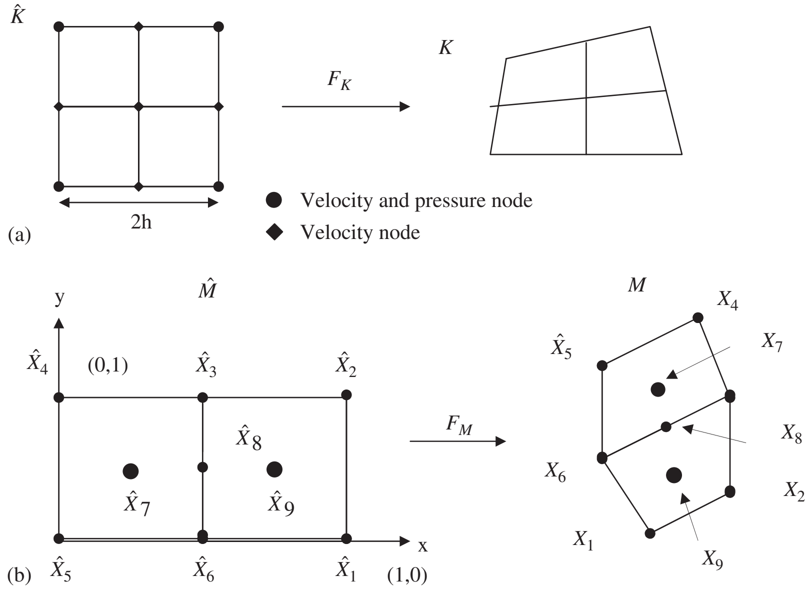

Figure 1. A two-dimensional macroelement partition.

Theorem 3
Suppose that M is a macroelement obtained by grouping two elements of ζ_{2h}. Then, the local null-space N_M is one dimensional, consisting of those functions which are constant on M.

Proof
Suppose that M is a union of two regular quadrilaterals as in Figure 1, then $\widehat{M}=\widehat{K}_1\cup\widehat{K}_2$, and $\widehat{K}_i=\bigcup_{j=1}^4 \widehat{K}_{ij}$, $\widehat{K}_{ij}\in\zeta_h$ for $i=1,2$. Further, we define a bilinear mapping $F_M:\widehat{M}\longrightarrow M$ such that $F_M=(F_1,F_2)$ is continuous on each $K_i\in\zeta_h$ and satisfies $F_M(\widehat{K}_i)=K_i$, where $\widehat{K}_i$ are the corresponding quadrilaterals shown in Figure 1.

Hence, for $\mathbf{u}\in\mathbf{V}_{0,M}$ and $p\in Q_M$

$$(\operatorname{div}\mathbf{u},p)_M=-(\mathbf{u},\nabla p)_M=-\sum_{i=1}^{2}\int_{\widehat{K}_i}\widehat{\mathbf{u}}^{\mathrm{T}}(\widehat{\mathbf{x}})\,J_F^{-\mathrm{T}}(\widehat{\mathbf{x}})\,\nabla\widehat{p}(\widehat{\mathbf{x}})\,|J_F(\widehat{\mathbf{x}})|\,\mathrm{d}\widehat{\mathbf{x}} \tag{7}$$

where $J_F(\widehat{\mathbf{x}})$ denotes the Jacobian matrix of F_M, $|J_F(\widehat{\mathbf{x}})|$ its determinant, and $J_F^{-\mathrm{T}}(\widehat{\mathbf{x}})$ the transpose matrix of $J_F^{-1}(\widehat{\mathbf{x}})$. Performing the necessary computation, we obtain

$$[\widehat{\mathbf{u}}^{\mathrm{T}}(\widehat{\mathbf{x}})J_F^{-\mathrm{T}}(\widehat{\mathbf{x}})\nabla\widehat{p}(\widehat{\mathbf{x}})|J_F(\widehat{\mathbf{x}})|]_{|\widehat{K}_{ij}}\in Q_2(\widehat{K}_{ij})$$

and

$$\widehat{\mathbf{u}}^{\mathrm{T}}(\widehat{\mathbf{x}})J_F^{-\mathrm{T}}(\widehat{\mathbf{x}})\nabla\widehat{p}(\widehat{\mathbf{x}})|J_F(\widehat{\mathbf{x}})| \text{ is continuous on } K_i,\ i=1,2$$

Hence, Simpson's composite rule gives the exact value of the integral in (7).

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We let $p\in N_M$ and choose $\mathbf{u}\in \mathbf{V}_{0,M}$ such that $\widehat{\mathbf{x}}_7(\frac{1}{2},\frac{1}{2})$, respectively $\widehat{\mathbf{x}}_9(\frac{3}{2},\frac{1}{2})$, is the only node at which $\widehat{\mathbf{u}}$ has nonvanishing degrees of freedom. Then, condition $(\operatorname{div}\mathbf{u},\, p)_M=0$ implies

$$J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_i)\nabla\widehat{p}(\widehat{\mathbf{x}}_i)|J_F(\widehat{\mathbf{x}}_i)|=0, \quad i=7,9 \tag{8}$$

Therefore,

$$\nabla\widehat{p}(\widehat{\mathbf{x}}_i)=0 \quad \text{for } i=7,9 \text{ since } |J_F(\widehat{\mathbf{x}})|\neq 0 \quad \forall\widehat{\mathbf{x}}\in\widehat{K}_i,\ \ i=1,2 \tag{9}$$

Setting $p_i=p(\mathbf{x}_i)=\widehat{p}(\widehat{\mathbf{x}}_i)$ for $i=1,2,\ldots,6$ and using (9), we obtain the relations

$$\begin{aligned} p_1&=p_3=p_5=\alpha\\ p_2&=p_4=p_6=\beta \end{aligned} \tag{10}$$

where α and β are two positive constants.

We then consider $\widehat{\mathbf{u}}\in\mathbf{V}_{0,M}$ such that $\widehat{\mathbf{x}}_8(1,\frac{1}{2})$ is the only node at which $\widehat{\mathbf{u}}$ has nonvanishing degrees of freedom. Hence, $(\operatorname{div}\mathbf{u},\, p)_M=0$ implies

$$\widehat{\mathbf{u}}^{\mathrm{T}}(\widehat{\mathbf{x}}_8)[(J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_8)\nabla\widehat{p}(\widehat{\mathbf{x}}_8)|J_F(\widehat{\mathbf{x}}_8))_{|\widehat{K}_1}+(J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_8)\nabla\widehat{p}(\widehat{\mathbf{x}}_8)|J_F(\widehat{\mathbf{x}}_8)|)_{|\widehat{K}_2}]=0 \tag{11}$$

Choosing $\mathbf{u}(\mathbf{x}_8)=\widehat{\mathbf{u}}(\widehat{\mathbf{x}}_8)=\overrightarrow{\widehat{\mathbf{x}}_6\widehat{\mathbf{x}}_3}$, a straightforward computation gives

$$J_F^{-1}(\widehat{\mathbf{x}}_8)\widehat{\mathbf{u}}(\widehat{\mathbf{x}}_8)|_{\widehat{K}_i}=[0,1]^{\mathrm{T}}, \quad i=1,2 \tag{12}$$

Since on $\widehat{K}_i$, $\widehat{p}_i(\widehat{x},\widehat{y})=a_i+b_i\widehat{x}+c_i\widehat{y}+d_i\widehat{x}\widehat{y}$, using (10) we get

$$\partial_2\widehat{p}(\widehat{\mathbf{x}}_8)_{|\widehat{K}_1}=c_1+d_1=p_5-p_4$$

$$\partial_2\widehat{p}(\widehat{\mathbf{x}}_8)_{|\widehat{K}_2}=c_2+d_2=p_5-p_4$$

where $\partial_l\equiv\partial/\partial x_l,\ l=1,d$,

$$\text{i.e.}\quad \partial_2\widehat{p}(\widehat{\mathbf{x}}_8)_{|\widehat{K}_1}=\partial_2\widehat{p}(\widehat{\mathbf{x}}_8)_{|\widehat{K}_2}\equiv\partial_2\widehat{p}(\widehat{\mathbf{x}}_8)=p_5-p_4 \tag{13}$$

Hence, from (11), (12), and (13) we obtain

$$\partial_2\widehat{p}(\widehat{\mathbf{x}}_8)[|J_F(\widehat{\mathbf{x}}_8)|_{|\widehat{K}_1}+|J_F(\widehat{\mathbf{x}}_8)|_{|\widehat{K}_2}]=0 \quad \text{i.e. } \partial_2\widehat{p}(\widehat{\mathbf{x}}_8)=0 \tag{14}$$

From which the required result follows

$$\text{i.e. } p_1=p_2=p_3=p_4=p_5=p_6 \qquad \square$$

4. TRILINEAR APPROXIMATION

We let ζ_{2h} be a regular partition of a three-dimensional region Ω into convex hexahedra and denote by ζ_h the partition obtained from ζ_{2h} by subdividing each K into eight hexahedra. We then define the approximate spaces

$$\begin{aligned} \mathbf{V}_h&=\{\mathbf{v}\in(H^1(\Omega)\cap C(\overline{\Omega}))^3\,|\,\mathbf{v}_{|K}\in(Q_1(K))^3,\ \forall K\in\zeta_h\}\\ Q_h&=\{q\in L_0^2(\Omega)\cap C(\overline{\Omega})\,|\,q_{|K}\in Q_1(K),\ \forall K\in\zeta_{2h}\} \end{aligned} \tag{15}$$

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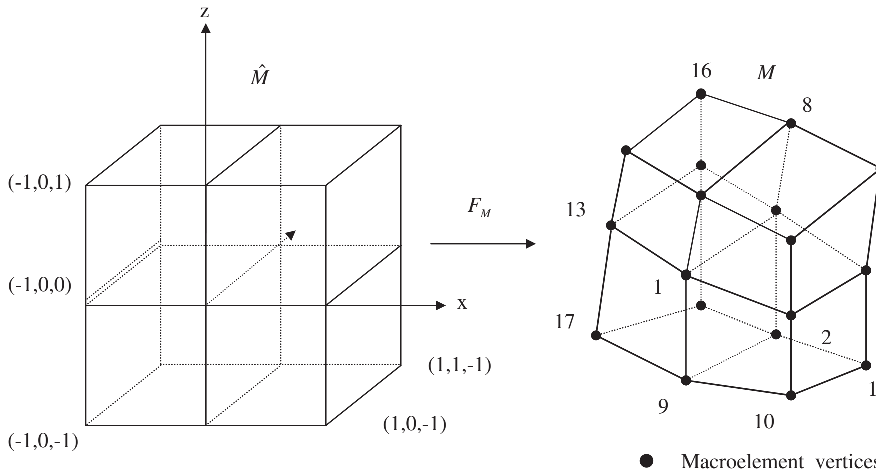

Figure 2. A three-dimensional macroelement partition. Coordinates of the reference macroelement vertices: $\hat{X}_1(0,0,0)$, $\hat{X}_2(1,0,0)$, $\hat{X}_3(1,1,0)$, $\hat{X}_4(0,1,0)$, $\hat{X}_5(0,0,1)$, $\hat{X}_6(1,0,1)$, $\hat{X}_7(1,1,1)$, $\hat{X}_8(0,1,1)$, $\hat{X}_9(0,0,-1)$, $\hat{X}_{10}(1,0,-1)$, $\hat{X}_{11}(1,1,-1)$, $\hat{X}_{12}(0,1,-1)$, $\hat{X}_{13}(-1,0,0)$, $\hat{X}_{14}(-1,1,0)$, $\hat{X}_{15}(-1,0,1)$, $\hat{X}_{16}(-1,1,1)$, $\hat{X}_{17}(-1,0,-1)$ and $\hat{X}_{18}(-1,1,-1)$.

Note that this mesh leads to one class of equivalent macroelements for which the hypotheses H1, H3, and H4 hold. Hence, to show that the above approximation is stable we need to check H2.

Theorem 4

Suppose that M is a macroelement obtained by grouping four elements of ζ_{2h}. Then, the local null-space N_M is one dimensional, consisting of those functions which are constant on M.

Proof

Suppose that M is a union of four regular hexahedra as in Figure 2, then $\widehat{M}=\bigcup_{i=1}^{4}\widehat{K}_i$, $K_i\in\zeta_{2h}$ and $\widehat{K}_i=\bigcup_{j=1}^{8}\widehat{K}_{ij}$, $\widehat{K}_{ij}\in\zeta_h$, for $i=\overline{1,4}$. Further, we define a trilinear mapping $F_M:\widehat{M}\longrightarrow M$ such that $F_M=(F_1,F_2,F_3)$ is a trilinear continuous mapping on each $K_i\in\zeta_h$ and satisfies $F_M(\widehat{K}_i)=K_i$, where $\widehat{K}_i$ are the corresponding cubes shown in Figure 2.

We then define the points $\widehat{\mathbf{x}}_A(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $\widehat{\mathbf{x}}_B(\frac{1}{2},\frac{1}{2},0)$, $\widehat{\mathbf{x}}_C(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, $\widehat{\mathbf{x}}_D(0,\frac{1}{2},\frac{1}{2})$, $\widehat{\mathbf{x}}_E(0,\frac{1}{2},-\frac{1}{2})$, $\widehat{\mathbf{x}}_F(-\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $\widehat{\mathbf{x}}_G(-\frac{1}{2},\frac{1}{2},0)$, $\widehat{\mathbf{x}}_H(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ and $\widehat{\mathbf{x}}_S(0,\frac{1}{2},0)$. Hence, for $\mathbf{u}\in\mathbf{V}_{0,M}$ and $p\in Q_M$ we have

$$(\operatorname{div}\mathbf{u},p)_M=-(\mathbf{u},\nabla p)_M=-\sum_{i=1}^{4}\int_{\widehat{K}_i}\widehat{\mathbf{u}}^{\mathrm{T}}(\widehat{\mathbf{x}})J_F^{-\mathrm{T}}(\widehat{\mathbf{x}})\nabla\widehat{p}(\widehat{\mathbf{x}})|J_F(\widehat{\mathbf{x}})|\,\mathrm{d}\widehat{\mathbf{x}} \tag{16}$$

Since F_M is trilinear on each $K_i\in\zeta_{2h}$ and the pressure is also trilinear, the computation of $J_F^{-\mathrm{T}}(\widehat{\mathbf{x}})\nabla\widehat{p}(\widehat{\mathbf{x}})|J_F(\widehat{\mathbf{x}})|$ gives

$$[J_F^{-\mathrm{T}}(\widehat{\mathbf{x}})\nabla\widehat{p}(\widehat{\mathbf{x}})|J_F(\widehat{\mathbf{x}})|]_{|\widehat{K}_i}=[R_1(\widehat{\mathbf{x}}),R_2(\widehat{\mathbf{x}}),R_3(\widehat{\mathbf{x}})]^{\mathrm{T}}\in(Q_2(\widehat{K}_i))^3 \tag{17}$$

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Moreover, since $\widehat{\mathbf{u}}(\widehat{\mathbf{x}})$ is trilinear on each $\widehat{K}_{ij}$ and continuous on $\widehat{K}_i$, we have

$$[\widehat{\mathbf{u}}^{\mathrm{T}}(\widehat{\mathbf{x}})J_F^{-\mathrm{T}}(\widehat{\mathbf{x}})\nabla\widehat{p}(\widehat{\mathbf{x}})|J_F(\widehat{\mathbf{x}})|]_{|\widehat{K}_i}\in Q_3(\widehat{K}_i)$$

Thus, Simpson's composite rule gives the exact value of the integral in (16).

We let $p\in N_M$ and choose $\widehat{\mathbf{u}}\in\mathbf{V}_{0,M}$ such that $\widehat{\mathbf{x}}_A$ (respectively $\widehat{\mathbf{x}}_C$, $\widehat{\mathbf{x}}_F$ and $\widehat{\mathbf{x}}_H$) is the only node at which $\widehat{\mathbf{u}}$ has nonvanishing degrees of freedom. Then, condition $(\operatorname{div}\mathbf{u}, p)_M=0$ implies

$$\nabla p(\widehat{\mathbf{x}})=\mathbf{0},\quad \widehat{\mathbf{x}}=\widehat{\mathbf{x}}_A, \widehat{\mathbf{x}}_C,\ \widehat{\mathbf{x}}_F \text{ and } \widehat{\mathbf{x}}_H \text{ since } |J_F(\widehat{\mathbf{x}})|\neq 0 \quad \forall\widehat{\mathbf{x}}\in K_i,\ \ i=\overline{1,4} \tag{18}$$

Choosing $\widehat{\mathbf{u}}\in\mathbf{V}_{0,M}$ such that $\widehat{\mathbf{x}}_B$ is the only node at which $\widehat{\mathbf{u}}$ has nonvanishing degrees of freedom, then $(\operatorname{div}\mathbf{u}, p)_M=0$ implies

$$(|J_F(\widehat{\mathbf{x}}_B)|J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_B)\nabla\widehat{p}(\widehat{\mathbf{x}}_B))_{|\widehat{K}_1}+(|J_F(\widehat{\mathbf{x}}_B)|J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_B)\nabla\widehat{p}(\widehat{\mathbf{x}}_B))_{|\widehat{K}_2}=0 \tag{19}$$

We denote by α^{ij}, $j=\overline{1,3}$, the column vectors of the matrix $(|J_F(\widehat{\mathbf{x}}_B)|J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_B))_{|\widehat{K}_i}$, $i=1,2$. Then, a straightforward calculation gives

$$\alpha^{13}=\alpha^{23}\quad\text{and}\quad \partial\widehat{p}(\widehat{\mathbf{x}}_B)_{|\widehat{K}_1}=\partial_1\widehat{p}(\widehat{\mathbf{x}}_B)_{|\widehat{K}_2}\equiv\partial_1\widehat{p}(\widehat{\mathbf{x}}_B) \tag{20}$$

Hence, (19) can be written as

$$[\alpha^{11}+\alpha^{21},\alpha^{12}+\alpha^{22},\alpha^{13}][\partial_1\widehat{p}(\widehat{\mathbf{x}}_B),\partial_2\widehat{p}(\widehat{\mathbf{x}}_B),\partial_3\widehat{p}(\widehat{\mathbf{x}}_B)_{|\widehat{K}_1}+\partial_3\widehat{p}(\widehat{\mathbf{x}}_B)_{|\widehat{K}_2}]^{\mathrm{T}}=0 \tag{21}$$

Due to the regularity of ζ_h, $(\alpha^{11}+\alpha^{21},\ \alpha^{12}+\alpha^{22},\ \alpha^{12}+\alpha^{22})$ is a nonsingular matrix. Hence, we get

$$\partial_l\widehat{p}(\widehat{\mathbf{x}}_B)=0,\quad l=1,2 \tag{22}$$

Repeating the same argument, respectively, for the macroelement interior points $\widehat{\mathbf{x}}_D$, $\widehat{\mathbf{x}}_E$ and $\widehat{\mathbf{x}}_G$, we obtain

$$\partial_l\widehat{p}(\widehat{\mathbf{x}}_D)=0,\ \ l=2,3;\quad \partial_l\widehat{p}(\widehat{\mathbf{x}}_E)=0,\ \ l=2,3;\quad\text{and}\quad \partial_l\widehat{p}(\widehat{\mathbf{x}}_G)=0,\ \ l=1,2 \tag{23}$$

Therefore, using (18), (22), and (23), we obtain

$$\begin{aligned} p_1=p_3=p_6=p_8=p_{10}=p_{12}=p_{14}=p_{15}=p_{17}=\alpha \\ p_2=p_4=p_5=p_7=p_9=p_{11}=p_{13}=p_{16}=p_{18}=\beta \end{aligned} \tag{24}$$

where $p_i=\widehat{p}(\widehat{\mathbf{x}}_i)$, $i=\overline{1,18}$; and α and β are two positive constants.

Thus, to complete the proof, we need to show that $\alpha=\beta$. Choose $\widehat{\mathbf{u}}\in\mathbf{V}_{0,M}$ such that $\widehat{\mathbf{x}}_S$ is the only node at which $\widehat{\mathbf{u}}$ has nonvanishing degrees of freedom. Then $(\operatorname{div}\mathbf{u}, p)_M=0$ implies

$$\begin{aligned}(J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_S)\nabla\widehat{p}(\widehat{\mathbf{x}}_S|J_F(\widehat{\mathbf{x}}_S)|)_{|\widehat{K}_1}+(J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_S)\nabla\widehat{p}(\widehat{\mathbf{x}}_S)|J_F(\widehat{\mathbf{x}}_S)|)_{|\widehat{K}_2}\\ +(J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_S)\nabla\widehat{p}(\widehat{\mathbf{x}}_S)|J_F(\widehat{\mathbf{x}}_S)|)_{|\widehat{K}_3}+(J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_S)\nabla\widehat{p}(\widehat{\mathbf{x}}_S)|J_F(\widehat{\mathbf{x}}_S)|)_{|\widehat{K}_4}=0\end{aligned} \tag{25}$$

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Denote by α^{ij} the columns of the matrix $(J_F^{-\mathrm{T}}(\widehat{\mathbf{x}}_S)|J_F(\widehat{\mathbf{x}}_S)|)_{|\widehat{K}_i}$, $i=\overline{1,4}$. Since on each $\widehat{K}_i$, $\widehat{p}_i(\widehat{x},\widehat{y},\widehat{z})=a_i+b_i\widehat{x}+c_i\widehat{y}+d_i\widehat{z}+e_i\widehat{x}\widehat{y}+f_i\widehat{x}\widehat{z}+g_i\widehat{y}\widehat{z}+h_i\widehat{x}\widehat{y}\widehat{z}$, then by similar arguments as above, we can show the following:

$$\begin{gathered}\alpha^{11}=\alpha^{31},\quad \alpha^{13}=\alpha^{23},\quad \alpha^{21}=\alpha^{41},\quad \alpha^{33}=\alpha^{43}\\ \partial_1\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_1}=\partial_1\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_2},\quad \partial_1\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_3}=\partial_1\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_4}\\ \partial_3\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_1}=\partial_3\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_3},\quad \partial_3\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_2}=\partial_3\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_4}\end{gathered}\tag{26}$$

and

$$\partial_2\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_1}=\partial_2\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_2}=\partial_2\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_3}=\partial_2\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_4}\equiv\partial_2\widehat{p}(\widehat{\mathbf{x}}_S)=p_4-p_1\tag{27}$$

Hence, using (26), (27), and (25), we obtain

$$\begin{bmatrix}\alpha^{11}+\alpha^{21}\\ \alpha^{12}+\alpha^{22}+\alpha^{32}+\alpha^{42}\\ \alpha^{13}+\alpha^{33}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\partial_1\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_1}+\partial_1\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_3}\\ \partial_2\widehat{p}(\widehat{\mathbf{x}}_S)\\ \partial_3\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_1}+\partial_3\widehat{p}(\widehat{\mathbf{x}}_S)_{|\widehat{K}_2}\end{bmatrix}=0\tag{28}$$

Due to the regularity of ζ_{2h}, $[\alpha^{11}+\alpha^{21},\alpha^{12}+\alpha^{22}+\alpha^{32}+\alpha^{42},\alpha^{13}+\alpha^{33}]$ is a nonsingular matrix and we obtain

$$\partial_2\widehat{p}(\widehat{\mathbf{x}}_S)=0\tag{29}$$

Thus, from (24), (27), and (29) follows the required result, i.e. N_M is one-dimensional consisting of functions that are constant on any macroelement M. □

Theorem 5

Suppose that problem (1) is approximated by regular finite elements as defined in (6) (respectively (15)). Then, for $\mathbf{u}\in(H^2(\Omega))^d\cap\mathbf{H}^1_{\mathbf{g}}(\Omega)$ and $p\in H^1(\Omega)$, with $\mathbf{H}^1_{\mathbf{g}}(\Omega)=\{\mathbf{v}\in(H^1(\Omega))^d\mid\mathbf{v}_{|\Gamma}=\mathbf{g}\}$, the following error estimate holds:

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega}\leqslant Ch(|\mathbf{u}|_{2,\Omega}+|p|_{1,\Omega})$$

Moreover, if Ω is convex in R^2 (or of class C^2 in R^3) we have the L^2-error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\leqslant Ch^2(|\mathbf{u}|_{2,\Omega}+|p|_{1,\Omega})$$

Proof

The proof of the first error estimate follows directly from the stability condition (4), and the L^2-velocity error estimate is obtained using the Aubin–Nische trick (see [1, 11]). □

5. NUMERICAL RESULTS

In this section, numerical results for two-dimensional Stokes flows, for the case $\nu=1$, are presented. The performance of the two-level bilinear approximation is compared with the biquadratic–bilinear

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Table I. Error estimates of the stable bilinear approximation.

h	$\|p-p_h\|_{0,\Omega}$	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$
$\frac{1}{4}$	0.510647	0.886496×10^{-2}	0.159219
$\frac{1}{8}$	0.258356	0.180605×10^{-2}	0.752692×10^{-1}
$\frac{1}{16}$	0.125986	0.406626×10^{-3}	0.367024×10^{-1}
$\frac{1}{32}$	0.660900×10^{-1}	0.970542×10^{-4}	0.181718×10^{-1}
$\frac{1}{64}$	0.333788×10^{-1}	0.238471×10^{-4}	0.905162×10^{-2}

one. Also, displayed are the velocity and pressure norms which confirm the convergence rates predicted by Theorem 3. For all problems, a preconditioned MINRES code is used to solve the global algebraic linear system.

5.1. *Test 1 problem*

As a first problem, we consider a flow in the unit square $[0, 1]\times[0, 1]$. The term $\mathbf{f}$ is chosen so that the solution is:

$$\mathbf{u}(x, y)=(u_x, u_y)^{\mathrm{T}}=(x^2, -2xy)^{\mathrm{T}} \quad \text{and} \quad p(x, y)=x^2+y^2+C$$

where C is a constant. Numerical results for this problem are displayed in Table I. The errors $\|p-p_h\|_{0,\Omega}$ and $\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$ converge at the predicted rates, while $\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$ converges faster than predicted. It is also observed that the method gives exact solution for the velocity components at the grid points. Also, the pressure solution plotted in Figure 3 indicate favourable comparison with the biquadratic–linear approximation. Further, the pressure plots predict the circular behaviour of the solution and do not present any wiggles that are found when using regularization and penalty type methods (see [17]).

5.2. *Test 2 problem*

The second problem consists in solving Stokes problem in the unit square $[0, 1]\times[0, 1]$, with exact solution:

$$\mathbf{u}(x, y)=(u_x, u_y)^{\mathrm{T}}, \quad p(x, y)=x^2-y^2$$

with

$$u_x=2x^2(1-x^2)y(1-y)(1-2y), \quad u_y=-2x(1-x)(1-2x)y^2(1-y^2)$$

Numerical results for this problem are displayed in Table II. The displayed results indicate that the errors $\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$ and $\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$ converge at the predicted rates, while $\|p-p_h\|_{0,\Omega}$ seems to converge at almost one degree higher than predicted. It is also observed, that eventhough the velocity solution is a fourth-degree polynomial, the numerical results obtained for both the velocity and pressure are comparable to the biquadratic solution as shown in Figure 4. This behaviour is believed to be due to the symmetry of the problem.

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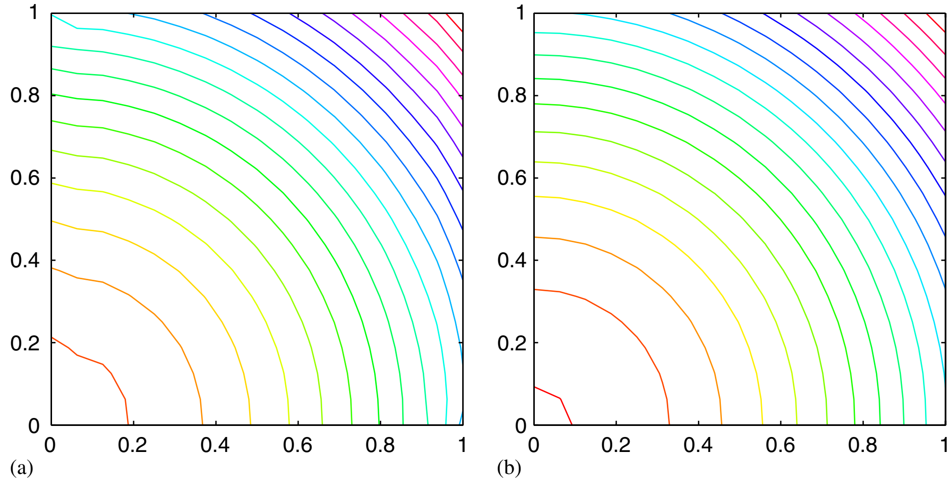

Figure 3. Pressure solution of test 1 problem on a 16×16 grid: (a) using a $Q1(h)$–$Q1(2h)$ approximation and (b) using a $Q2$–$Q1$ approximation.

Table II. Error estimates of the stable bilinear approximation.

h	$\|p - p_h\|_{0,\Omega}$	$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}$
$\frac{1}{4}$	0.263523×10^{-1}	0.243569×10^{-2}	0.304875×10^{-1}
$\frac{1}{8}$	0.757418×10^{-2}	0.609685×10^{-3}	0.156370×10^{-1}
$\frac{1}{16}$	0.204403×10^{-2}	0.151712×10^{-3}	0.777448×10^{-2}
$\frac{1}{32}$	0.578548×10^{-3}	0.375650×10^{-4}	0.387329×10^{-2}
$\frac{1}{64}$	0.162971×10^{-3}	0.925248×10^{-5}	0.193278×10^{-2}

5.3. Lid-driven cavity flow

The third test problem is that of lid-driven cavity flow on the domain $[-1, 1] \times [-1, 1]$. Our aim here is to assess the performance of the two-level bilinear approximation using a graded mesh near $x = -1$, $x = 1$, $y = -1$, and $y = 1$. We impose a regularized boundary condition, that is $u_x(-1, y) = u_x(1, y) = u_x(x, -1) = 0$ and $u_x(x, 1) = 1 - x^4$; for $-1 \leqslant x \leqslant 1$. Numerical pressure solutions obtained for a 16×16 grid, using the two-level bilinear and the biquadratic–bilinear approximations are displayed in Figure 5. Clearly, Figure 5 indicates that the two solutions are comparable. Further, the streamlines are computed from the velocity solution, by solving Poisson's equation subject to a zero boundary condition, and are displayed in Figure 6. The latter illustrates the recirculation of the flow at bottom corners which is consistent with what has been reported by many researchers (see [18]).

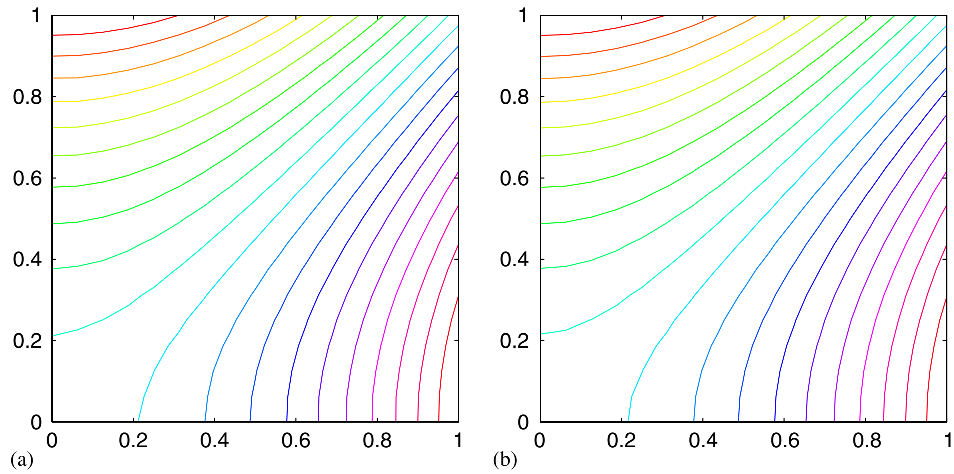

Figure 4. Pressure solution of test 2 problem on a 16×16 grid: (a) using a $Q1(h)$–$Q1(2h)$ approximation and (b) using a $Q2$–$Q1$ approximation.

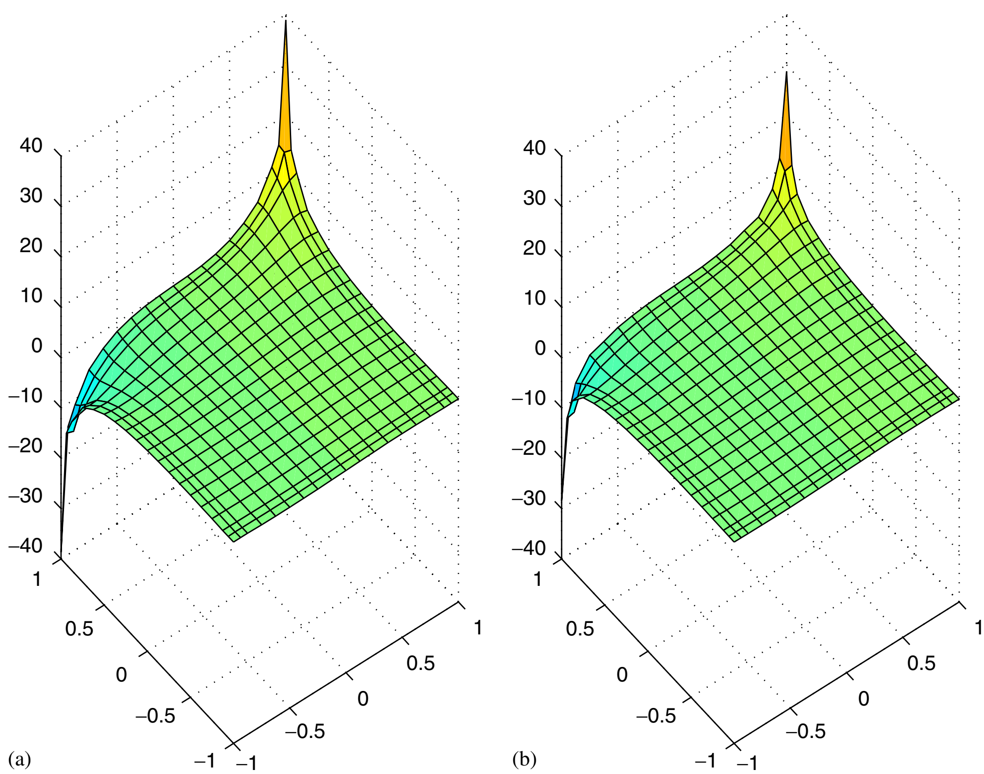

Figure 5. Pressure solution on a 16×16 grid: (a) current approach and (b) biquadratic approximation.

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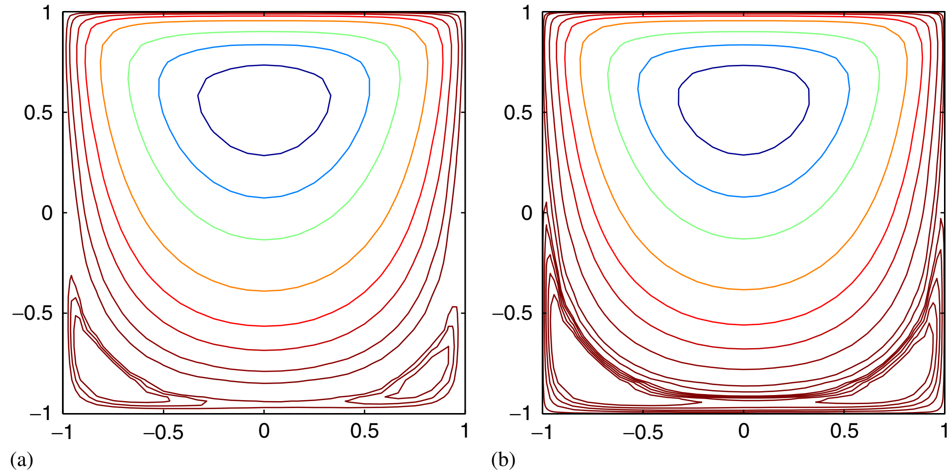

Figure 6. Streamline plot and secondary recirculation at the bottom corners of the cavity domain: (a) current approach and (b) using a $Q2$–$Q1$ approximation.

6. CONCLUSION

In this paper, we have analysed the stability of some low-order approximations for fluid flow problems. The use of these elements is attractive because of the simplicity in handling the element properties and this is expected to be useful for three-dimensional problems. The numerical results for the two-level bilinear approximation clearly confirm the convergence rates predicted by the theory and compare favourably with some standard finite elements.

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